// THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. VII. JANUARY, 1900. No. 1.

THE THEORY OF MATHEMATICAL INFERENCE.

By G. J. STOKES, Professor of Logic and Philosophy in Queen's College, Cork, Ireland.

One of the simplest theories of mathematical inference and by no means the least plausible is that involved in the extreme Nominalist doctrine that the conclusion of a train of reasoning is but a restatement in changed language of the original data from which we start, a renaming in short, of what had been otherwise expressed. Nowhere has this theory as applied to mathematical inference been more clearly put forward than by James Mill in his "Analysis of the Phenomena of the Human Mind."

"The Predications of Arithmetic" he says are another instance of the same thing. 'One and one are two.' This again is a mere process of naming. What I call one and one, in numbering things, are objects, sensations, or clusters of sensations; suppose, the striking of the clock. The same sounds which I call one and one I call also two; I have for these sensations, therefore, two names which are exactly equivalent: so when I say, one and one and one are three: or when I say, two and two are four: ten and ten are twenty: and the same when I put together any two numbers whatsoever. The series of thoughts in these instances is merely a series of names applicable to the same thing, and meaning the same thing.

Besides the two purposes of language, of which I took notice at the beginning of this inquiry; the recording of a man's thoughts for his own use, and the communication of them to others; there is a use, to which language is subservient, of which some account is yet to be given. These are complex sensations,

and complex ideas, made up of so many items, that one is not distinguishable from another. Thus, a figure of one hundred sides, is not distinguishable from one of ninety-nine. A thousand men in a crowd are not distinguishable from nine hundred and ninety-nine. But in all cases, in which the complexity of the idea arises from the repetition of the same idea, names can be invented upon a plan, which shall render them distinct, up to the very highest degree of complication. Numbers are a set of names contrived upon this plan, and for this very purpose. Ten and the numbers below ten, are the repetition of so many ones: twenty, thirty, forty, etc., up to a hundred, are the repetition of so many tens: two hundred, three hundred, etc., the repetition of so many hundreds, and so on. These are names, which afford an immediate reference to the ones or units, of which they are composed; and the highest numbers are as easily distinguished by the difference of a unit as the lowest. All the processes of Arithmetic are only so many contrivances to substitute a distinct name for an indistinct one. What, for example, is the purpose of addition? Suppose I have six numbers, of which I desire to take the sum, 18, 14, 9, 25, 19, 15; these names, eighteen, and fourteen, and nine, etc., form a compound name; but a name which is not distinct. By summing them up, I get another name, exactly equivalent, one hundred, which is in the highest degree distinct, and gives me an immediate reference to the units or items of which it is composed; and this is of the highest utility.

That the Predications of Geometry are of the same nature with those of Arithmetic, is a truth of the greatest importance, and capable of being established by very obvious reasoning. It is well known, that all reasoning about quantity can be expressed in the form of algebraic equations. But the two sides of an algebraic equation are of necessity two marks or two names for the same thing; of which the one on the right-hand side is more distinct, at least to the present purpose of the inquirer, than the one on the left-hand side; and the whole purpose of an algebraic investigation, which is a mere series of changes of names, is to obtain, at last, a distinct name, a name the marking power of which is perfectly known to us, on the right-hand side of the equation. The language of geometry itself, in the more simple cases, makes manifest the same observation. The amount of the three angles of a triangle, is twice a right angle. I arrive at this conclusion, as it is called, by a process of reasoning: that is to say, I find out a name "twice a right angle," which much more distinctly points out to me a certain quantity, than my first name, "amount of the three angles of a triangle;" and the process by which I arrive at this name is a successive change of names, and nothing more; as any one may prove to himself by merely observing the steps of the demonstration."

It is easy to criticize the doctrine of this passage. Strictly taken it would reduce all mathematical reasoning to a series of identities—a mere change of names. If we admit however that total and partial identities—relations of wholes and parts—may be expressed by such changes, the doctrine coincides with the opinion of those who consider that the fundamental truths of mathemat-

ics are of an analytical not a synthetical character. This opinion has not only received the support of Leibnitz as well as of the school of philosophy to whom the truths of mathematics—especially those of Arithmetic and Algebra—represent merely verbal propositions, but it has been peculiarly strengthened by recent mathematical developments. If we consult the writings of those who have maintained the opposite doctrine, viz., the synthetical character of mathematical truths we shall find that they maintain this only of certain of the fundamental truths employed in mathematics. Thus the axioms proper of Euclid are admitted by Kant and Mansel to be analytical. When they seek for an example of a synthetical truth, they find it in the fifth and sixth postulates, sometimes enumerated as the eleventh and twelfth axioms. But it is precisely such principles as the latter which are now regarded as expressing not universal and necessary principles of all geometry but only the particular and contingent properties of the space with which we are acquainted; and of which Clifford asserts "for all we know, any or all of them may be false." If now by mathematical inference or reasoning be understood the form of reasoning common to all mathematical thought, there seems to be left as a residuum, only those processes of analytical inference which are expounded in the ordinary formal or Aristotelian logic.

It is possible to support this view, to regard those principles which are really synthetic and fertile in mathematics, as either gathered from actual experience, or as hypothetically assumed in regard to some possible experience; and, on the other hand to regard the process by which these fundamental assumptions are worked out into their consequences as purely syllogistic. The necessity with which these consequences flowed would then be strictly formal and logical. This view would appear to be in exact accordance with the general principles laid down by J. S. Mill on the subject of demonstration and necessary truths (Logic, Book II, Chapter VI). It differs from Mill's view, only in not regarding the axioms proper as inductions from experience, and in extending the postulates to embrace those possibilities of Non-Euclidean geometry which Mill did not contemplate. Notwithstanding some criticism of Mill, this I understand to be the view actually adopted by Clifford. A theory more completely agreeing with Mill's principles is put forward by Erdmann in his work 'Die Axiome der Geometrie' resting his conclusions on the investigations of Riemann and Helmholtz.

It is to be observed however that this doctrine is in complete opposition to the theory contained in the quotation from James Mill with which we started. On that theory the rich content of mathematics could be evolved by a series of analytical or verbal transformations. J. S. Mill in the chapter of the Logic from which we have quoted, clearly shows the impossibility of such a view. Mathematical inference leads to new truths, and new truths, according to Mill, can only be reached if inference be impregnated by experience. Hence Mill held that axioms and postulates come from this source.

It still remains to be asked, Can Formal Logic, Syllogism or mere verbal inference, perform the attenuated task left to it, viz., the inferring process?

Can verbal propositions, if no longer competent to give a new content, nevertheless be the means of passing from one content to another? At first sight, the great resemblance between the elementary propositions of mathematics and the propositions of Formal Logic, seems to favour this view. When in Arithmetic we form the judgment $2+1=3$, or in Geometry discover the quantitative equivalence of the three angles of a triangle to two right angles, the resemblance to such purely analytical relations as are formulated by the *dictum de omni et nullo* is very great. In fact, our theory reduces predication itself to the equation of groups. Nevertheless, this resemblance is, I believe illusory, and so far is mere verbal inference from being able to perform the function, which James Mill allotted to it, that it cannot even perform the more modest task reserved to it by J. S. Mill.

It has long been a matter of observation that inferences exist, which while perfectly rigorous, yet do not admit of syllogistic analysis. The argument *a fortiori* is an example, and many others are furnished by what has been called the logic of relatives. These are examples of what older logicians called material consequence. The device by which Mansel and others have tried to reduce them to syllogistic form, is, as De Morgan truly says, an evasion. Both kinds of inference are found in mathematics but that inference which is most peculiarly mathematical is of the second kind, which escapes or defies the analysis of Formal Logic. Mill has exhibited Euclid I, 5 in syllogistic form but the reduction of the reasoning to this form is purely external. The force of mathematical reasoning is independent of the reduction. This is already implied in Dugald Stewart's remark, referred to by Mill, that it is not necessary to our seeing the conclusiveness of the proof in mathematical reasonings, that the axioms should be expressly adverted to. The same thing is conceded, perhaps unwittingly by Hamilton. "Mathematical, like all other reasoning," he says, "is syllogistic; but here *the perspicuous necessity of the matter necessitates the correctness of the form:* we cannot reason wrong."

If, now, we have reason to believe, that there exists in mathematical inference, a "necessity of the matter" existing in itself, and not merely derivative from the logical form, the question arises: Can we isolate it for itself? If we can do this we shall have grounds for concluding, that what is really fruitful in mathematics, is not, as has been so often supposed, initial definitions, axioms, etc., which we afterwards logically analyze and develop, but a certain synthetic mode of inference not identical with the analytical inference of formal logic, but distinct from it, *sui generis*, and perhaps opposed in character.

Before passing to the consideration of this synthetic and material necessity, it may be well to point out, that the whole controversy as it has hitherto existed between the *a priori* and *a posteriori* schools, between the Kantian and the empiricist, becomes for us irrelevant. In a paper in Mind in 1884 I pointed out that the Kantian theory does not explain the synthesis in mathermatical truths. It only places the synthesis finished and complete, in the subject. Clifford puts the same point in another way when he says that the Kantian theory

makes the general statements of mathematics into particular statements. On the other hand the empirical school makes no attempt to explain the necessity of the consequence in itself. It is this which we are seeking to isolate. When this isolation has been effected, it will appear that mathematical inference is so related to ordinary analytical reasoning, as to stand in need neither of the Kantian nor experience hypothesis, neither of *a priori* form nor of inseparable association, in order that its necessary and universal character may be accounted for.

The idea has sometimes been entertained, that logical analysis has reached through the calculus contained in Boole's Laws of Thought this more penetrating character and that here we have an instrument which can make inferences beyond the range of formal logic. This is not so. Mr. A. J. Ellis has shown that it really does less; and in the papers of Leslie Ellis it is pointed out that for dealing with those forms of collateral inferences which we find in tha logic of relatives it is as ineffectual as formal logic.

The immense suggestiveness of Boole's work lies in the circumstance that he has reduced formal logic to a calculus and that logical doctrines are put in a form in which they suggest mathematical analogues. But the suggestion is, for the most part one of opposition. The things are brought into the same plane and thereby their opposition becomes apparent. This is precisely what from the foregoing we should expect. If, now, taking Boole's Laws of Thought as an exposition of formal logic in mathematical form, we ask the question 'Can we find within the area of mathematics any calculus presenting that antithetic but complementary character which a form of synthetic inference should present, as contrasted with a form of analytical inference?' I think we may answer 'Yes.' If we compare the fundamental equations of Boole's Laws of Thought with the equations which characterize some of those forms of multiplication discussed by Grassmann, and employed by him in his 'Ausdehnungslehre' we seem to find the antithesis which we seek. Already in the principles of the Differential Calculus this antithesis was to be found, and by an intuition of genius was perceived by Boole. For the purpose of this paper we shall confine ourselves to the following equations.

In Boole's Laws of Thought we find

$$x^2 = x \quad \text{.................(1).}$$

$$x(1-x) = 0 \quad \text{.................(2).}$$

In Grassmann's we find

$$e_p{}^2 = 0 \quad \text{.................(3).}$$

$$e_p e_s = -e_s e_p \quad \text{.................(4).}$$

If we compare these two sets of equations we shall find that they differ in this fundamental characteristic, that whereas the first set of equations belong to an algebra of self-identical unrelated units the second set belong to an algebra in which relation, synthesis, references beyond self, is essential.

The equations (3) and (4) have given rise to some difference of opinion in

regard to their mutual relations. H. Hankel, H. J. S. Smith and Mr. Whitehead assert that the first of these equations follows necessarily from the second. Buchheim however points out that this is not the case, that (4) does not involve (3), and Clifford regards (4) as following from (3); that is, we may assume (4), without asserting (3), but not (3) without (4).

In this Clifford and Buchheim seem to be right. The equation $r_s r_p = -r_p r_s$ proves $r_s^2 = 0$ (in the case where $p = s$) if r_s and r_p exist only as necessary relatives which become equal to 0 when the difference and therefore the relation between them disappears. If they exist as such, equation (4) seems to follow, but equation (3) follows from this relativity, not from (4), which may be affirmed even when the factors involved exist out of and apart from their mutual relation. The truth seems to be that equations of this type of algebra may participate in characters derived from equations of the Boole type, and so give rise to hybrid species which may be useful for particular interpretations. In general we may distinguish (1) the purely logical algebra whose relations only of identity and difference, coincidence and exclusion are admitted. (2) The algebra of material consequence where the units are not indifferent but where the *entia* involved are essentially relative and the algebra consequently synthetic.

Both these algebras lead to a final equation in which each passes over into the opposite. Boole's $x(1-x)=0$ is in his system the ultimate condition of logical interpretability but it also expresses, the one relation into which logical terms enter with each other, and the equation $i_s^2 = \pm 1$ (Clifford) expresses the disappearance of relativity and the return of the merely logical relations of difference and identity.

Ordinary Arithmetic, Algebra, and Metrical Geometry are a combination of these two kinds of Algebra. Up to a certain point, the principles of both can be alternately applied, but at one point the necessity arises for a deeper fusion. The introduction of relation into a logical calculus involves, as has been pointed out by Mr. Venn, the very thing which Boole excludes—the admission of exponents. Conversely, in ordinary mathematics the appearance of exponents involves essential relation. In the sign $\pm\sqrt{}$ the alternation of the purely logical and the relational aspects is still continued, and the same is the case in the Differential Calculus. But in the latter, and in imaginary expressions this alternation of independent aspects ceases. It the calculus the externality of the logical consideration ceases at infinity. In the imaginary it ceases in the immediate combination of the signs $+$ and $-$

In a paper on the Imaginary of Logic (British Association, 1898) I put forward the view that as the square root of a positive quantity is $+$ *or* $-$ the square root of a negative quantity may be expected to be $+$ *and* $-$, in view of the logical relation between '*and*' and '*or*' pointed out originally by De Morgan, and subsequently, and independently by Schroeder. This theory is the opposite of one put forward in an early number of the Cambridge and Dublin Mathematical Journal by Gregory, viz., That the signs $+$ and $-$ are themselves the subject of the exponential operation. The object of the paper was to show that

a necessary relation of the signs + and — as affecting the factors respectively was the essential characteristic of imaginary quantities. The essence of this relation would then coincide with Boole's $x(1-x)$. The subsequent portion of the paper sought to verify this theory throughout the various geometrical interpretations which imaginaries have received.

Since the paper was written I have found that its conclusions receive support from a remarkable series of papers by Mr. A. B. Kempe F. R. S. Mr. Kempe has shown that between the mathematical theory of points and the logical theory of statements, a striking correspondence exists. Between the laws defining the form of a system of points, and those defining the form of a system of statements, perfect sameness exists with one exception. The former is subject to a law to which the latter is not subject. It is sufficient here to say that it is the law "which expresses the fact that two straight lines can only cut once."

From these conclusions we may draw the converse inference, that the laws which govern geometrical theory can be deduced from logical or purely analytical principles, taken in conjunction with that law in which the form of a system of points differs from the form of a system of statements. We have now to ask, Is there anything omitted from the form of a system of statements as contemplated by Mr. Kempe, or by the ordinary logic (and there is complete agreement between them) which would account for the absence of the particular law which distinguishes geometrical theory? I think there is. Mr. Kempe in order to effect his assimilation of the logical to the geometrical theory, and in particular in explaining the processes of immediate inference has introduced two constants which play the same part in the logical theory that the 'absolute' does in geometry. He entitles them 'truism' and 'falsism' respectively. It is by relation to these that such logical relations as contrariety, sub-contrariety, sub-alternation analogous to the metrical relations of points in geometry are determined. He considers "truisms" and "falsisms" as propositions or statements standing in the system of statements on the same footing with all other statements. In reality this is not so. The truism and falsism of Mr. Kempe are really the laws of Identity and Contradiction in disguise, and every synthetic statement or proposition expresses more than what these laws require. The principle that a real proposition refers to, or is a synthesis with, something more than itself, is as old as Aquinas, and is indeed the fundamental principle which makes our thinking dependent on experience (Cf. Bradley's Principles of Logic). It is the non-recognition of this which prevents Mr. Kempe from evolving the relation of non-collinearity from the relation of a truism and falsism to each other which ought to be capable of being done, if it were true that these propositions could rank *pari passu* with all other propositions. A truism is not as such a true proposition. Apart from the postulate of synthesis no logical relation exists between the truism and falsism. Contradictories are in this case compatible as Venn and Kant before him have pointed out.

If these views be true I believe it to be possible to deduce the properties of Euclidean space, not from the analytical laws of thought, but from the pure

postulate of synthesis, when subjected to conditions arising from these laws. The postulate can be shown to involve two things (1) Infinity, (2) the necessary relation or connection of what Mr. Kempe styles truism and falsism equivalent to Boole's $x(1-x)=0$.

It remains to point out the connection which exists between the logical 'absolute' of Mr. Kempe and the theory of the imaginary referred to in the course of this paper. I was led to that analysis from consideration of the correspondence between the logical relation of a copulative or conjunctive to a disjunctive proposition, and the mathematical relation of imaginary roots to the roots of positive quantities. A similar relation has been perceived by Mr. Kempe. "The symmetrical resultant of the triad [of statements] a, b, f [f=falsism] is the statement usually written, a and b, and the symmetrical resultant of the triad, a, b, t [t=truism] is the statement usually written a or b." If the relation of truism and falsism or in Boole's language $x(1-x)$ be, as we assert the essence of the mathematical imaginary; and if the same constants have in Mr. Kempe's analysis disclosed themselves as the essence of the geemetrical 'absolute' a deep-lying relation is revealed between the methods of metrical and projective Geometry.

Finally, there exists a curious analogy between the geometrical, and certain theories of the metaphysical absolute. The temptation lies near at hand to evolve the synthetical or given element out of the laws of Identity and Contradiction. Fichte's evolution of the Non-Ego out of the Ego is effected in this way and thus arises his theory of the absolute Ego. Precisely the some error is committed, if an attempt be made to dispense with the postulate of synthesis, with the *given*, and to evolve mathematics out of analytical propositions. Mr. Kempe comes near this mistake when he treats truisms and falsisms as propositions on a line with all other propositions.

It remains to draw the final conclusions of this paper. The fertile propositions of mathematics from which its wealth of content and the treasures of mathematical knowledge are drawn, are not synthetical in the sense in which Kant and the Empiricists alike maintain them to be, viz., that the truths pre-exist and are thus seen to be synthetical, the synthetical character being as it were something subsequent to the content of the proposition and attaching to it as it were adjectivally; but in this sense that those propositions are themselves the product of pure synthesis, that the very possibility of advance from entity to entity or unit to unit, or relation to correlate, determines all those laws which mathematics is employed in exploring and tracing into all their consequences, and which are infinitely more fruitful than the analytical laws of Formal Logic or the Calculus of classes and statements. Pure synthesis generally is that "necessity of the matter" of which Hamilton spoke, the principal of material consequences, which characterizes every genuine department of mathematics and defies further logical analysis.

Fancourt, Balbriggan, Ireland, 10 August, 1899.